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REPRESENTING SUMS OF FINITE PRODUCTS OF CHEBYSHEV POLYNOMIALS OF THE SECOND KIND AND FIBONACCI POLYNOMIALS IN TERMS OF CHEBYSHEV POLYNOMIALS

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ABSTRACT. In this paper, we will consider sums of finite products of Chebyshev polynomials of the second kind and Fibonacci polynomials. Then we represent each of those sums of finite products in terms of the four kinds of Chebyshev polynomials which involve the Gauss hypergeometric function ${}_2F_1$.

1. Introduction and preliminaries

For any nonnegative integer n, the falling factorial polynomials $(x)_n$ and the rising factorial polynomials $< x >_n$ are respectively given by

$$(x)_n = x(x-1)\cdots(x-n+1),\ (n \geq 1),\ (x)_0 = 1, \tag{1.1}$$

$$< x >_n = x(x+1)\cdots(x+n-1),\ (n \geq 1),\ < x >_0 = 1. \tag{1.2}$$

The Gauss hypergeometric function ${}_2F_1(a, b; c; x)$ are defined by

$${}_2F_1(a, b; c; x) = \sum_{n=0}^{\infty} \frac{< a >_n < b >_n}{< c >_n} \frac{x^n}{n!},\ (|x| < 1). \tag{1.3}$$

As to the classical orthogonal polynomials we only need some basic knowledge about Chebyshev polynomials which we will recall here in below. For full accounts for this fascinating area of mathematics, the interested reader may refer to [2,3,13].

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The Chebyshev polynomials of the first, second, third and fourth kinds, and Fibonacci polynomials are respectively defined by the following recurrence relations.

$$T_{n+2}(x) = 2xT_{n+1}(x) - T_n(x),\ (n \geq 0),\ T_0(x) = 1,\ T_1(x) = x, \tag{1.4}$$

$$U_{n+2}(x) = 2xU_{n+1}(x) - U_n(x),\ (n \geq 0),\ U_0(x) = 1,\ U_1(x) = 2x, \tag{1.5}$$

$$V_{n+2}(x) = 2xV_{n+1}(x) - V_n(x),\ (n \geq 0),\ V_0(x) = 1,\ V_1(x) = 2x - 1, \tag{1.6}$$

$$W_{n+2}(x) = 2xW_{n+1}(x) - W_n(x),\ (n \geq 0),\ W_0(x) = 1, W_1(x) = 2x + 1, \tag{1.7}$$

$$F_{n+2}(x) = xF_{n+1}(x) + F_n(x),\ (n \geq 0),\ F_0(x) = 0,\ F_1(x) = 1. \tag{1.8}$$

In terms of generating functions, they are respectively given by

$$\frac{1 - xt}{1 - 2xt + t^2} = \sum_{n=0}^{\infty} T_n(x)t^n, \tag{1.9}$$

$$F(t, x) = \frac{1}{1 - 2xt + t^2} = \sum_{n=0}^{\infty} U_n(x)t^n, \tag{1.10}$$

$$\frac{1 - t}{1 - 2xt + t^2} = \sum_{n=0}^{\infty} V_n(x)t^n, \tag{1.11}$$

$$\frac{1 + t}{1 - 2xt + t^2} = \sum_{n=0}^{\infty} W_n(x)t^n, \tag{1.12}$$

$$G(t, x) = \frac{1}{1 - xt - t^2} = \sum_{n=0}^{\infty} F_{n+1}(x)t^n. \tag{1.13}$$

The Chebyshev polynomials of the first, second, third and fourth kinds, and the Fibonacci polynomials are explicitly expressed respectively by the following.

$$\begin{aligned} T_n(x) &= {}_2F_1(-n, n; \tfrac{1}{2}; \tfrac{1-x}{2}) \\ &= \frac{n}{2} \sum_{l=0}^{[\frac{n}{2}]} (-1)^l \frac{1}{n-l} \binom{n-l}{l} (2x)^{n-2l},\ (n \geq 1), \end{aligned} \tag{1.14}$$

$$\begin{aligned} U_n(x) &= (n+1){}_2F_1(-n, n+2; \tfrac{3}{2}; \tfrac{1-x}{2}) \\ &= \sum_{l=0}^{[\frac{n}{2}]} (-1)^l \binom{n-l}{l} (2x)^{n-2l},\ (n \geq 1), \end{aligned} \tag{1.15}$$

$$V_n(x) = {}_2F_1(-n, n+1; \tfrac{1}{2}; \tfrac{1-x}{2}) = \sum_{l=0}^{n} \binom{n+l}{2l} 2^l (x-1)^l, \ (n \geq 0), \tag{1.16}$$

$$W_n(x) = (2n+1){}_2F_1(-n, n+1; \tfrac{3}{2}; \tfrac{1-x}{2}) = (2n+1) \sum_{l=0}^{n} \frac{2^l}{2l+1} \binom{n+l}{2l} (x-1)^l, \ (n \geq 0), \tag{1.17}$$

$$F_{n+1}(x) = \sum_{l=0}^{[\frac{n}{2}]} \binom{n-l}{l} x^{n-2l}, \ (n \geq 0). \tag{1.18}$$

It is well known that the Chebyshev polynomials are also given by the Rodrigues' formulas.

$$T_n(x) = \frac{(-1)^n 2^n n!}{(2n)!} (1-x^2)^{\frac{1}{2}} \frac{d^n}{dx^n} (1-x^2)^{n-\frac{1}{2}}, \tag{1.19}$$

$$U_n(x) = \frac{(-1)^n 2^n (n+1)!}{(2n+1)!} (1-x^2)^{-\frac{1}{2}} \frac{d^n}{dx^n} (1-x^2)^{n+\frac{1}{2}}, \tag{1.20}$$

$$(1-x)^{-\frac{1}{2}} (1+x)^{\frac{1}{2}} V_n(x) = \frac{(-1)^n 2^n n!}{(2n)!} \frac{d^n}{dx^n} (1-x)^{n-\frac{1}{2}} (1+x)^{n+\frac{1}{2}}, \tag{1.21}$$

$$(1-x)^{\frac{1}{2}} (1+x)^{-\frac{1}{2}} W_n(x) = \frac{(-1)^n 2^n n!}{(2n)!} \frac{d^n}{dx^n} (1-x)^{n+\frac{1}{2}} (1+x)^{n-\frac{1}{2}}. \tag{1.22}$$

Probably, the most important characteristic of Chebyshev polynomials is their orthogonalities with respect to various weight functions which are given by the following.

$$\int_{-1}^{1} (1-x^2)^{-\frac{1}{2}} T_n(x) T_m(x) dx = \frac{\pi}{\mathcal{E}_n} \delta_{n,m}, \tag{1.23}$$

$$\int_{-1}^{1} (1-x^2)^{\frac{1}{2}} U_n(x) V_m(x) dx = \frac{\pi}{2} \delta_{n,m}, \tag{1.24}$$

$$\int_{-1}^{1} \left(\frac{1+x}{1-x}\right)^{\frac{1}{2}} V_n(x) V_m(x) dx = \pi \delta_{n,m}, \tag{1.25}$$

$$\int_{-1}^{1} \left(\frac{1-x}{1+x}\right)^{\frac{1}{2}} W_n(x) W_m(x) dx = \pi \delta_{n,m}, \tag{1.26}$$

where

$$\mathcal{E}_n = \begin{cases} 1, & \text{if } n = 0, \\ 2, & \text{if } n \geq 1, \end{cases} \qquad \delta_{n,m} = \begin{cases} 0, & \text{if } n \neq m, \\ 1, & \text{if } n = m. \end{cases} \tag{1.27}$$

Let us put

$$\alpha_{n,r}(x) = \sum_{i_1+i_2+\cdots+i_{r+1}=n} U_{i_1}(x)U_{i_2}(x)\cdots U_{i_{r+1}}(x),\ (n, r \geq 0), \tag{1.28}$$

$$\beta_{n,r}(x) = \sum_{i_1+i_2+\cdots+i_r=n} F_{i_1+1}(x)F_{i_2+1}(x)\cdots F_{i_r+1}(x),\ (n \geq 0, r \geq 1). \tag{1.29}$$

Note here that both $\alpha_{n,r}(x)$ and $\beta_{n,r}(x)$ have degree n.

In this paper, we will consider the sums of finite products of Chebyshev polynomials of the second kind in (1.28) and those of Fibonacci polynomials in (1.29). Then we will express each of $\alpha_{n,r}(x)$ and $\beta_{n,r}(x)$ as linear combinations of the four kinds of Chebyshev polynomials $T_n(x), U_n(x), V_n(x)$, and $W_n(x)$. They are found by explicit computations and using the general formulas in Propositions 3 and 4. They can be derived by making use of orthogonalities, Rodrigues' formulas and integration by parts.

Our results are as follows.

Theorem 1. *Let n, r be integers with $n \geq 0, r \geq 1$. Then we have the following identities.*

$$\sum_{i_1+i_2+\cdots+i_{r+1}=n} U_{i_1}(x)U_{i_2}(x)\cdots U_{i_{r+1}}(x)$$

$$= \frac{1}{r!}\sum_{j=0}^{[\frac{n}{2}]} \mathcal{E}_{n-2j}\binom{r+j}{r}(n-j+r)_r T_{n-2j}(x) \tag{1.30}$$

$$= \frac{1}{r!}\sum_{j=0}^{[\frac{n}{2}]} (n-2j+1)\binom{r+j-1}{r-1}(n-j+r)_{r-1} U_{n-2j}(x) \tag{1.31}$$

$$= \frac{1}{r!}\sum_{j=0}^{n} \binom{r+[\frac{j}{2}]}{r}(n-[\tfrac{j}{2}]+r)_r V_{n-j}(x) \tag{1.32}$$

$$= \frac{1}{r!}\sum_{j=0}^{n} (-1)^j\binom{r+[\frac{j}{2}]}{r}(n-[\tfrac{j}{2}]+r)_r W_{n-j}(x). \tag{1.33}$$

Here $[x]$ denotes the greatest integer $\leq x$.

Theorem 2. *Let n, r be integers with $n \geq 0, r \geq 1$. Then the following identities hold true.*

$$\sum_{i_1+i_2+\cdots+i_r=n} F_{i_1+1}(x)F_{i_2+1}(x)\cdots F_{i_r+1}(x)$$

$$= \frac{\binom{n+r-1}{n}}{2^n}\sum_{j=0}^{[\frac{n}{2}]}\mathcal{E}_{n-2j}\binom{n}{j}{}_2F_1(-j,j-n;1-n-r;-4)T_{n-2j}(x) \tag{1.34}$$

$$= \frac{\binom{n+r-1}{n}}{2^n(n+1)}\sum_{j=0}^{[\frac{n}{2}]}(n-2j+1)\binom{n+1}{j}{}_2F_1(-j,j-n-1;1-n-r;-4)U_{n-2j}(x) \tag{1.35}$$

$$= \frac{\binom{n+r-1}{n}}{2^n}\sum_{j=0}^{n}\binom{n}{[\frac{j}{2}]}{}_2F_1(-[\tfrac{j}{2}],[\tfrac{j}{2}]-n;1-n-r;-4)V_{n-j}(x) \tag{1.36}$$

$$= \frac{\binom{n+r-1}{n}}{2^n}\sum_{j=0}^{n}(-1)^j\binom{n}{[\frac{j}{2}]}{}_2F_1(-[\tfrac{j}{2}],[\tfrac{j}{2}]-n;1-n-r;-4)W_{n-j}(x). \tag{1.37}$$

Sums of finite products of Bernoulli, Euler and Genocchi polynomials have been treated in [1,8,9]. In particular, they are expressed in terms of Bernoulli polynomials by deriving Fourier series expansions for the functions closely related to those sums of finite products. Also, the same were done for the sums of finite products $\alpha_{n,r}(x)$ and $\beta_{n,r}(x)$ in (1.28) and (1.29) in [7]. For other applications of Chebyshev polynomials, one might want to look at [4,10].

2. Proof of Theorem 1

Here in this section we will prove Theorem 1. For this purpose, we first state two results that will be used in showing Theorems 1 and 2.

The results (a) and (b) in Proposition 3 are respectively from the equations (24) and (36) of [6], while (c) and (d) are stated respectively in the equations (23) and (38) of [5]. All of them can be easily derived from the orthogonality relations in (1.23)-(1.26) and the Rodrigues' formulas in (1.19)-(1.22).

Proposition 3. *Let $q(x) \in \mathbb{R}[x]$ be a polynomial of degree n. Then we have the following.*

$$
\begin{aligned}
(a)\quad & q(x)=\sum_{k=0}^{n} C_{k,1}T_k(x),\ \ where\\
& C_{k,1}=\frac{(-1)^k 2^k k!\mathcal{E}_k}{(2k)!\pi}\int_{-1}^{1} q(x)\frac{d^k}{dx^k}(1-x^2)^{k-\frac{1}{2}}dx.\\
(b)\quad & q(x)=\sum_{k=0}^{n} C_{k,2}U_k(x),\ \ where\\
& C_{k,2}=\frac{(-1)^k 2^{k+1}(k+1)!}{(2k+1)!\pi}\int_{-1}^{1} q(x)\frac{d^k}{dx^k}(1-x^2)^{k+\frac{1}{2}}dx.\\
(c)\quad & q(x)=\sum_{k=0}^{n} C_{k,3}V_k(x),\ \ where\\
& C_{k,3}=\frac{(-1)^k k! 2^k}{(2k)!\pi}\int_{-1}^{1} q(x)\frac{d^k}{dx^k}(1-x)^{k-\frac{1}{2}}(1+x)^{k+\frac{1}{2}}dx.\\
(d)\quad & q(x)=\sum_{k=0}^{n} C_{k,4}W_k(x),\ \ where\\
& C_{k,4}=\frac{(-1)^k k! 2^k}{(2k)!\pi}\int_{-1}^{1} q(x)\frac{d^k}{dx^k}(1-x)^{k+\frac{1}{2}}(1+x)^{k-\frac{1}{2}}dx.
\end{aligned}
$$

We note here that, for $k=0$, the following integrals are the moments of the four kinds of Chebyshev polynomials.

Proposition 4. *For any nonnegative integers m and k, we have the following.*

$$
\begin{aligned}
(a)\quad & \int_{-1}^{1}(1-x^2)^{k-\frac{1}{2}}x^m dx\\
& =\begin{cases} 0, & if\ m\equiv 1\ (mod\ 2),\\ \frac{m!(2k)!\pi}{2^{m+2k}(\frac{m}{2}+k)!(\frac{m}{2})!k!}, & if\ m\equiv 0\ (mod\ 2).\end{cases}\\
(b)\quad & \int_{-1}^{1}(1-x^2)^{k+\frac{1}{2}}x^m dx\\
& =\begin{cases} 0, & if\ m\equiv 1\ (mod\ 2),\\ \frac{m!(2k+2)!\pi}{2^{m+2k+2}(\frac{m}{2}+k+1)!(\frac{m}{2})!(k+1)!}, & if\ m\equiv 0\ (mod\ 2).\end{cases}
\end{aligned}
$$

$$(c)\quad \int_{-1}^{1}(1-x)^{k-\frac{1}{2}}(1+x)^{k+\frac{1}{2}}x^m dx$$
$$=\begin{cases}\frac{(m+1)!(2k)!\pi}{2^{m+2k+1}(\frac{m+1}{2}+k)!(\frac{m+1}{2})!k!}, & \text{if } m\equiv 1 \ (mod\ 2),\\ \frac{m!(2k)!\pi}{2^{m+2k}(\frac{m}{2}+k)!(\frac{m}{2})!k!}, & \text{if } m\equiv 0\ (mod\ 2).\end{cases}$$

$$(d)\quad \int_{-1}^{1}(1-x)^{k+\frac{1}{2}}(1+x)^{k-\frac{1}{2}}x^m dx$$
$$=\begin{cases}-\frac{(m+1)!(2k)!\pi}{2^{m+2k+1}(\frac{m+1}{2}+k)!(\frac{m+1}{2})!k!}, & \text{if } m\equiv 1 \ (mod\ 2),\\ \frac{m!(2k)!\pi}{2^{m+2k}(\frac{m}{2}+k)!(\frac{m}{2})!k!}, & \text{if } m\equiv 0\ (mod\ 2).\end{cases}$$

Proof.

$$(a)\quad \int_{-1}^{1}(1-x^2)^{k-\frac{1}{2}}x^m dx=\frac{1}{2}(1+(-1)^m)\int_0^1(1-y)^{k+\frac{1}{2}-1}y^{\frac{m+1}{2}-1}dy, \quad (2.1)$$

which is 0 for m odd. So we assume that m is even. Then (2.1) is

$$B(k+\tfrac{1}{2},\tfrac{m}{2}+\tfrac{1}{2})=\frac{\Gamma(k+\frac{1}{2})\Gamma(\frac{m}{2}+\frac{1}{2})}{\Gamma(k+\frac{m}{2}+1)}=\frac{(2k)!\Gamma(\frac{1}{2})m!\Gamma(\frac{1}{2})}{2^{2k}k!2^m(\frac{m}{2})!(k+\frac{m}{2})!}.$$

(b) The follows from (a) by replacing k by $k+1$.
(c) The follows from (a) and (b) by noting

$$\int_{-1}^{1}(1-x)^{k-\frac{1}{2}}(1+x)^{k+\frac{1}{2}}x^m dx$$
$$=\int_{-1}^{1}(1-x^2)^{k-\frac{1}{2}}x^m dx+\int_{-1}^{1}(1-x^2)^{k-\frac{1}{2}}x^{m+1}dx.$$

(d) Similarly to (c), this follows from (a) and (b). □

As was shown in [15] and mentioned in [12], we can show the following lemma by differentiating the equation (1.10).

Lemma 5. *Let n, r be nonnegative integers. Then we have the following identity.*

$$\sum_{i_1+i_2+\cdots+i_{r+1}=n}U_{i_1}(x)U_{i_2}(x)\cdots U_{i_{r+1}}(x)=\frac{1}{2^r r!}U_{n+r}^{(r)}(x), \quad (2.2)$$

where the sum is over all nonnegative integers $i_1, i_2, \cdots, i_{r+1}$, with $i_1+i_2+\cdots+i_{r+1}=n$.

From (1.15), the rth derivative of $U_n(x)$ is given by

$$U_n^{(r)}(x)=\sum_{l=0}^{[\frac{n-r}{2}]}(-1)^l\binom{n-l}{l}(n-2l)_r 2^{n-2l}x^{n-2l-r}. \quad (2.3)$$

In particular, we have

$$U_{n+r}^{(r+k)}(x) = \sum_{l=0}^{[\frac{n-k}{2}]} (-1)^l \binom{n+r-l}{l} (n+r-2l)_{r+k} 2^{n+r-2l} x^{n-k-2l}. \tag{2.4}$$

Here we will prove only (1.31) and (1.33) in Theorem 1, as (1.30) and (1.32) can be shown analogously to (1.31) and (1.33) respectively.

As in (1.28), we put

$$\alpha_{n,r}(x) = \sum_{i_1+i_2+\cdots+i_{r+1}=n} U_{i_1}(x)U_{i_2}(x)\cdots U_{i_{r+1}}(x),$$

and let

$$\alpha_{n,r}(x) = \sum_{k=0}^{n} C_{k,2} U_k(x). \tag{2.5}$$

Then, from (b) of Proposition 3, (2.2), (2.4), and integrating by parts k times, we have

$$\begin{aligned} C_{k,2} &= \frac{(-1)^k 2^{k+1}(k+1)!}{(2k+1)!\pi} \int_{-1}^{1} \alpha_{n,r}(x) \frac{d^k}{dx^k}(1-x^2)^{k+\frac{1}{2}} dx \\ &= \frac{(-1)^k 2^{k+1}(k+1)!}{(2k+1)!\pi 2^r r!} \int_{-1}^{1} U_{n+r}^{(r)}(x) \frac{d^k}{dx^k}(1-x^2)^{k+\frac{1}{2}} dx \\ &= \frac{2^{k+1}(k+1)!}{(2k+1)!\pi 2^r r!} \int_{-1}^{1} U_{n+r}^{(r+k)}(x)(1-x^2)^{k+\frac{1}{2}} dx \\ &= \frac{2^{k+1}(k+1)!}{(2k+1)!\pi 2^r r!} \sum_{l=0}^{[\frac{n-k}{2}]} (-1)^l \binom{n+r-l}{l} (n+r-2l)_{r+k} 2^{n+r-2l} \\ &\quad \times \int_{-1}^{1} (1-x^2)^{k+\frac{1}{2}} x^{n-k-2l} dx. \end{aligned} \tag{2.6}$$

From (2.6), (b) of Proposition 4, and after some simplifications, we obtain

$$\begin{aligned} C_{k,2} &= \frac{1}{r!} \sum_{l=0}^{[\frac{n-k}{2}]} (-1)^l \binom{n+r-l}{l} (n+r-2l)_{r+k} \\ &\quad \times \begin{cases} 0, & \text{if } k \not\equiv n \pmod 2, \\ \frac{(k+1)(n-k-2l)!}{(\frac{n+k}{2}-l+1)!(\frac{n-k}{2}-l)!}, & \text{if } k \equiv n \pmod 2. \end{cases} \end{aligned} \tag{2.7}$$

Before proceeding further, we recall Chu-Vandermonde formula given by

$$ {}_2F_1(-n,a;c;1)=\frac{<c-a>_n}{<c>_n},\ (c\neq 0,-1,\cdots,1-n). \tag{2.8}$$

Now, from (2.5), and (2.7), we have

$$\begin{aligned}
\alpha_{n,r}(x)&=\frac{1}{r!}\sum_{\substack{0\le k\le n\\ k\equiv n\ (mod 2)}}\sum_{l=0}^{[\frac{n-k}{2}]}(-1)^l\binom{n+r-l}{l}(n+r-2l)_{r+k}\\
&\quad\times\frac{(k+1)(n-k-2l)!}{(\frac{n+k}{2}-l+1)!(\frac{n-k}{2}-l)!}U_k(x)\\
&=\frac{1}{r!}\sum_{j=0}^{[\frac{n}{2}]}(n-2j+1)U_{n-2j}(x)\sum_{l=0}^{j}\frac{(-1)^l(n+r-l)!}{(n-j-l+1)!(j-l)!l!}\\
&=\frac{1}{r!}\sum_{j=0}^{[\frac{n}{2}]}\frac{(n-2j+1)(n+r)!}{(n-j+1)!j!}U_{n-2j}(x)\sum_{l=0}^{j}\frac{<-j>_l<j-n-1>_l}{<-n-r>_l\ l!}\\
&=\frac{1}{r!}\sum_{j=0}^{[\frac{n}{2}]}\frac{(n-2j+1)(n+r)!}{(n-j+1)!j!}{}_2F_1(-j,j-n-1;-n-r;1)U_{n-2j}(x).
\end{aligned}\tag{2.9}$$

From (2.8) and (2.9), we finally obtain

$$\begin{aligned}
\alpha_{n,r}(x)&=\frac{1}{r!}\sum_{j=0}^{[\frac{n}{2}]}\frac{(n-2j+1)(n+r)!<-r-j+1>_j}{(n-j+1)!j!<-n-r>_j}U_{n-2j}(x)\\
&=\frac{1}{r!}\sum_{j=0}^{[\frac{n}{2}]}\frac{(n-2j+1)(n+r)!(r+j-1)_j}{(n-j+1)!j!(n+r)_j}U_{n-2j}(x)\\
&=\frac{1}{r!}\sum_{j=0}^{[\frac{n}{2}]}(n-2j+1)\binom{r+j-1}{r-1}(n-j+r)_{r-1}U_{n-2j}(x).
\end{aligned}\tag{2.10}$$

This shows (1.31) and we can prove (1.30) similarly.

Remark 6. *Applying* (1.30) *and* (1.31) *to the sth derivatives* $T_n^{(s)}(x)$ *and* $U^{(s)}(x)$, *we can show that*

$$T_n^{(s)}(x) = n2^{s-1} \sum_{j=0}^{[\frac{n-s}{2}]} \mathcal{E}_{n-s-2j} \binom{s+j-1}{s-1} (n-j-1)_{s-1} T_{n-s-2j}(x), \ (s \geq 1), \tag{2.11}$$

$$U_n^{(s)}(x) = 2^s \sum_{j=0}^{[\frac{n-s}{2}]} (n-s-2j+1) \binom{s+j-1}{s-1} (n-j)_{s-1} U_{n-s-2j}(x), \ (s \geq 1). \tag{2.12}$$

This agrees with the results in [11].

Next, we show the equation (1.33). We let

$$\alpha_{n,r}(x) = \sum_{k=0}^{n} C_{k,4} W_k(x). \tag{2.13}$$

Then, from (d) of Proposition 3, (2.2), (2.4), and integrating by parts k times, we get

$$\begin{aligned} C_{k,4} = & \frac{k!2^k}{(2k)!\pi 2^r r!} \sum_{l=0}^{[\frac{n-k}{2}]} (-1)^l \binom{n+r-l}{l} (n+r-2l)_{r+k} \\ & \times 2^{n+r-2l} \int_{-1}^{1} (1-x)^{k+\frac{1}{2}} (1+x)^{k-\frac{1}{2}} x^{n-k-2l} dx. \end{aligned} \tag{2.14}$$

From (2.14), (d) of Proposition 4, and after some simplifications, we obtain

$$\begin{aligned} C_{k,4} = & \frac{1}{r!} \sum_{l=0}^{[\frac{n-k}{2}]} (-1)^l \frac{(n+r-l)!}{l!} \\ & \times \begin{cases} -\frac{n-k-2l+1}{2(\frac{n+k+1}{2}-l)!(\frac{n-k+1}{2}-l)!}, & \text{if } k \not\equiv n \pmod 2, \\ \frac{1}{(\frac{n+k}{2}-l)!(\frac{n-k}{2}-l)!}, & \text{if } k \equiv n \pmod 2. \end{cases} \end{aligned} \tag{2.15}$$

Combining (2.15) and (2.13), and invoking (2.8), we now obtain

$$\begin{aligned}\alpha_{n,r}(x) &= \frac{1}{r!}\sum_{j=0}^{[\frac{n}{2}]} W_{n-2j}(x)\sum_{l=0}^{j}\frac{(-1)^l(n+r-l)!}{l!(n-j-l)!(j-l)!} \\ &\quad - \frac{1}{r!}\sum_{j=0}^{[\frac{n-1}{2}]} W_{n-1-2j}(x)\sum_{l=0}^{j}\frac{(-1)^l(n+r-l)!}{l!(n-j-l)!(j-l)!} \\ &= \frac{1}{r!}\sum_{j=0}^{[\frac{n}{2}]}\binom{r+j}{r}(n-j+r)_r W_{n-2j}(x) \\ &\quad - \frac{1}{r!}\sum_{j=0}^{[\frac{n-1}{2}]}\binom{r+j}{r}(n-j+r)_r W_{n-1-2j}(x) \\ &= \frac{1}{r!}\sum_{j=0}^{n}(-1)^j\binom{r+[\frac{j}{2}]}{r}(n-[\tfrac{j}{2}]+r)_r W_{n-j}(x).\end{aligned} \tag{2.16}$$

This shows (1.33) and (1.32) can be proved similarly.

3. Proof of Theorem 2

In this section, we will show (1.34) and (1.36) of Theorem 2, as (1.35) and (1.37) can be proved analogously to (1.34) and (1.36) respectively.

We start with the following lemma which is stated as the equation (7) in [14].

Lemma 7. *Let n, r be integers with $n \geq 0$, $r \geq 1$. Then we have the following identity.*

$$\sum_{i_1+i_2+\cdots+i_r=n} F_{i_1+1}(x)F_{i_2+1}(x)\cdots F_{i_r+1}(x) = \frac{1}{(r-1)!}F_{n+r}^{(r-1)}(x), \tag{3.1}$$

where the sum is over all nonnegative integers $i_1, i_2, \cdots, i_r$, with $i_1+i_2+\cdots+i_r = n$.

From (1.18), we note that the rth derivative of $F_{n+1}(x)$ is given by

$$F_{n+1}^{(r)}(x) = \sum_{l=0}^{[\frac{n-r}{2}]}\binom{n-l}{l}(n-2l)_r x^{n-r-2l}. \tag{3.2}$$

Especially, we have

$$F_{n+r}^{(r+k-1)}(x) = \sum_{l=0}^{[\frac{n-k}{2}]}\binom{n+r-l-1}{l}(n+r-2l-1)_{r+k-1}x^{n-k-2l}. \tag{3.3}$$

As in (1.29), we put

$$\beta_{n,r}(x) = \sum_{i_1+i_2+\cdots+i_r=n} F_{i_1+1}(x)F_{i_2+1}(x)\cdots F_{i_r+1}(x),$$

and let

$$\beta_{n,r}(x) = \sum_{k=0}^{n} C_{k,1}T_k(x). \tag{3.4}$$

Then, from (a) of Proposition 3, (3.1), (3.3), and integrating by parts k times, we have

$$\begin{aligned} C_{k,1} &= \frac{(-1)^k 2^k k! \mathcal{E}_k}{(2k)!\pi} \int_{-1}^{1} \beta_{n,r}(x) \frac{d^k}{dx^k}(1-x^2)^{k-\frac{1}{2}}dx \\ &= \frac{(-1)^k 2^k k! \mathcal{E}_k}{(2k)!\pi(r-1)!} \int_{-1}^{1} F_{n+r}^{(r-1)}(x) \frac{d^k}{dx^k}(1-x^2)^{k-\frac{1}{2}}dx \\ &= \frac{2^k k! \mathcal{E}_k}{(2k)!\pi(r-1)!} \int_{-1}^{1} F_{n+r}^{(r+k-1)}(x)(1-x^2)^{k-\frac{1}{2}}dx \\ &= \frac{2^k k! \mathcal{E}_k}{(2k)!\pi(r-1)!} \sum_{l=0}^{[\frac{n-k}{2}]} \binom{n+r-l-1}{l} (n+r-2l-1)_{r+k-1} \\ &\quad \times \int_{-1}^{1} (1-x^2)^{k-\frac{1}{2}} x^{n-k-2l} dx. \end{aligned} \tag{3.5}$$

From (3.5), (a) of Proposition 4, and after some simplifications, we obtain

$$\begin{aligned} C_{k,1} = {} & \frac{\mathcal{E}_k}{(r-1)!2^n} \sum_{l=0}^{[\frac{n-k}{2}]} \frac{(n+r-l-1)!}{l!} \\ & \times \begin{cases} 0, & \text{if } k \not\equiv n \pmod 2, \\ \frac{4^l}{(\frac{n+k}{2}-l)!(\frac{n-k}{2}-l)!}, & \text{if } k \equiv n \pmod 2. \end{cases} \end{aligned} \tag{3.6}$$

Combing (3.4), and (3.6), we now have

$$
\begin{aligned}
\beta_{n,r}(x) &= \frac{1}{2^n(r-1)!} \sum_{\substack{0\le k\le n \\ k\equiv n\ (mod 2)}} \mathcal{E}_k \sum_{l=0}^{[\frac{n-k}{2}]} \frac{(n+r-l-1)!4^l}{(\frac{n+k}{2}-l)!(\frac{n-k}{2}-l)!l!} T_k(x) \\
&= \frac{1}{2^n(r-1)!} \sum_{j=0}^{[\frac{n}{2}]} \mathcal{E}_{n-2j} \sum_{l=0}^{j} \frac{(n+r-l-1)!4^l}{(n-l-j)!(j-l)!l!} T_{n-2j}(x) \\
&= \frac{1}{2^n(r-1)!} \sum_{j=0}^{[\frac{n}{2}]} \mathcal{E}_{n-2j} \frac{(n+r-1)!}{(n-j)!j!} T_{n-2j}(x) \sum_{l=0}^{j} \frac{(n-j)_l(j)_l 4^l}{(n+r-1)_l l!} \\
&= \frac{1}{2^n(r-1)!} \sum_{j=0}^{[\frac{n}{2}]} \mathcal{E}_{n-2j} \frac{(n+r-1)!}{(n-j)!j!} T_{n-2j}(x) \\
&\quad \times \sum_{l=0}^{j} \frac{<-j>_l<j-n>_l}{<1-n-r>_l} \frac{(-4)^l}{l!} \\
&= \frac{\binom{n+r-1}{n}}{2^n} \sum_{j=0}^{[\frac{n}{2}]} \binom{n}{j} \mathcal{E}_{n-2j} {}_2F_1(-j, j-n; 1-n-r; -4) T_{n-2j}(x).
\end{aligned} \tag{3.7}
$$

This shows (1.34) and (1.35) can be proved similarly.
Next, we would like to prove the equation (1.36). For this, we let

$$
\beta_{n,r}(x) = \sum_{k=0}^{n} C_{k,3} V_k(x). \tag{3.8}
$$

Then, from (c) of Proposition 3, (3.1), (3.3), and integrating by parts k times, we get

$$
\begin{aligned}
C_{k,3} &= \frac{k!2^k}{(2k)!\pi(r-1)!} \sum_{l=0}^{[\frac{n-k}{2}]} \binom{n+r-l-1}{l} (n+r-2l-1)_{r+k-1} \\
&\quad \times \int_{-1}^{1} (1-x)^{k-\frac{1}{2}} (1+x)^{k+\frac{1}{2}} x^{n-k-2l} dx.
\end{aligned} \tag{3.9}
$$

From (3.9), (c) of Proposition 4, and after some simplification, we obtain

$$\begin{aligned} C_{k,3} &= \frac{1}{2^n(r-1)!}\sum_{l=0}^{[\frac{n-k}{2}]}\frac{(n+r-l-1)!}{l!} \\ &\times \begin{cases} \frac{(n-k-2l+1)2^{2l-1}}{(\frac{n+k+1}{2}-l)!(\frac{n-k+1}{2}-l)!}, & \text{if } k \not\equiv n \pmod 2, \\ \frac{2^{2l}}{(\frac{n+k}{2}-l)!(\frac{n-k}{2}-l)!}, & \text{if } k \equiv n \pmod 2. \end{cases} \end{aligned} \tag{3.10}$$

Combining (3.8) and (3.10), we now have

$$\begin{aligned} \beta_{n,r}(x) &= \frac{1}{2^n(r-1)!}\sum_{j=0}^{[\frac{n-1}{2}]}V_{n-1-2j}(x)\sum_{l=0}^{j}\frac{(n+r-l-1)!4^l}{(n-l-j)!(j-l)!l!} \\ &+ \frac{1}{2^n(r-1)!}\sum_{j=0}^{[\frac{n}{2}]}V_{n-2j}(x)\sum_{l=0}^{j}\frac{(n+r-l-1)!4^l}{(n-l-j)!(j-l)!l!} \\ &= \frac{\binom{n+r-1}{n}}{2^n}\sum_{j=0}^{[\frac{n-1}{2}]}\binom{n}{j}{}_2F_1(-j,j-n;1-n-r;-4)V_{n-1-2j}(x) \\ &+ \frac{\binom{n+r-1}{n}}{2^n}\sum_{j=0}^{[\frac{n}{2}]}\binom{n}{j}{}_2F_1(-j,j-n;1-n-r;-4)V_{n-2j}(x) \\ &= \frac{\binom{n+r-1}{n}}{2^n}\sum_{j=0}^{n}\binom{n}{[\frac{j}{2}]}{}_2F_1(-[\frac{j}{2}],[\frac{j}{2}]-n;1-n-r;-4)V_{n-j}(x) \end{aligned} \tag{3.11}$$

This shows (1.36) and we can prove (1.37) similarly.

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